Physical Properties and Processes: Scaling☆

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Introduction

The physical properties of natural porous media, such as soils and sediments, typically exhibit a high degree of inherent spatial variability over the landscape which greatly complicates attempts to predict their behavior using continuum-scale mathematical models based on partial differential equations. This complexity is reduced, however, if values of the properties at different locations can be connected to one another by scale factors defined in such a way that the partial differential equation governing the behavior of the properties has exactly the same mathematical form at all locations in spite of spatial variability. Inherent spatial heterogeneity is thus accounted for by considering physical properties at one location to be merely scaled versions — either magnified or reduced — of those at another location. Whenever this simplification through scaling is possible, the governing differential equation and the physical laws which underlie it are said to exhibit "scale invariance." Scale invariance, however, turns out to require that any non-constant coefficients in the governing differential equation be restricted to certain functions of the dependent variable, which then also must be the same at all locations.

Despite this restriction, such uniform functional relationships have often been observed in controlled field experiments, supporting the applicability of a scale-invariant partial differential equation for describing the behavior of natural porous media. This concept will be illustrated using a concrete example, the Richards equation for one-dimensional infiltration, which describes volumetric water content changes in space and time as water moves downward in a soil profile. Scale invariance of this partial differential equation requires that the functional relationships between its two transport coefficients, water diffusivity and hydraulic conductivity, and the volumetric water content be power laws. This condition, in turn, leads to experimentally testable relationships among the scale factors.

Understanding Scale Invariance

The physical properties of natural soils are inherently variable when observed from place to place. Volumetric water content — to choose perhaps the simplest example — can vary substantially over the span of an agricultural field, even if only the surface horizon of the soil is sampled and care has been taken to provide a uniform input of water prior to sampling. The complexity that this kind of spatial variability would bring to any attempt to understand and predict the temporal evolution of volumetric water content across an entire field is evident, and the same conclusion follows for other important soil physical properties.

Now suppose that the underlying cause of this spatial variability in soil physical properties has to do with the inherent heterogeneity of soil texture or morphology, but not with the fundamental mechanism by which soil physical properties change in response to inputs of matter and energy. If this supposition is true, then the underlying physical laws that govern the behavior of these properties must be uniform across a field, even if the properties themselves are not inherently uniform. Moreover, the functional relationships between the properties (e.g., the relationship between volumetric water content and matric potential) should also be uniform, in the sense that each such relationship is quantified by the same parametric equation everywhere in a field, although with different values of its parameters at different places owing to spatial heterogeneity.

That spatial uniformity in the physical laws describing the behavior of soil properties implies spatial uniformity in the functional relationships between these properties may not seem obvious, but this conclusion follows once the physical laws are expressed mathematically as partial differential equations. Imposing a condition of spatial uniformity on the form of a governing partial differential equation places a strong constraint on the mathematical form of the relationships between its dependent variable and any non-constant coefficients that appear in the differential equation, with the result that these relationships necessarily must also be uniform.

To make these ideas concrete, we shall consider the non-linear partial differential equation that governs the behavior of the volumetric water content under isothermal, isohaline, isochoric infiltration in soil, known as the Richards equation:

☆*Change History:* December 2015. G Sposito thoroughly revised the text throughout. The reference list was updated to include recent review articles and books.

$$\partial\theta/\partial t = \partial[D(\theta)\partial\theta/\partial z]/\partial z - (\partial K/\partial\theta)(\partial\theta/\partial z) \quad [1]$$

where θ (L^3 L^{-3}) is volumetric water content, $t(T)$ is time, $z(L)$ is vertical position, D(θ) (L^2T^{-1}) is the water diffusivity, dependent on θ, and K(θ) (LT^{-1}) is the hydraulic conductivity, also dependent on θ. Equation [1] describes the time evolution of the volumetric water content in terms of two empirical transport coefficients, D(θ) and K(θ), whose relationship to θ must be specified in order to apply the differential equation in a predictive manner. Once this has been done, one would like to apply Eq. [1] to model (one-dimensional) infiltration in a field, despite the expectation that D(θ) and K(θ) may exhibit widely-varying values because of inherent spatial variability in the field. Irrespective of this variability, D(θ) and K(θ) each is to preserve the same functional relationship to θ everywhere.

We now assume that the inherent spatial heterogeneity of field soils in the horizontal plane can be expressed mathematically by means of three scale factors that connect values of the dependent and independent variables in Eq. [1] at any two places:

$$\theta' = \mu\theta,\ t' = \delta t,\ z' = \alpha z \quad [2]$$

where μ, δ, and α are the positive-valued scale factors. These scale factors relate $\theta'(t', z')$ at one place to $\theta(t, z)$ at another, under the premise that the governing physical law in Eq. [1] is the same at both places and all of the spatial variability in the volumetric water content across a field beyond that stemming from the differential equation can be represented by scale factors. Thus, at the "primed" location, the sole effect of inherent soil heterogeneity is to magnify or reduce the volumetric water content by a scale factor μ relative to the volumetric water content at the "un-primed" location. Moreover, in respect to description of the space-time dependence of the volumetric water content as given by the physical laws epitomized in Eq. [1], time and space themselves in the "primed" location are magnified or reduced by the scale factors δ and α, respectively, relative to their values at the "un-primed" location. In this sense, soil water behavior at the "primed" location is assumed to be simply a scaled version of its behavior at the "un-primed" location.

Since Eq. [1] must have the same mathematical form at both the "un-primed" and "primed" locations, it must look just the same after it is transformed by multiplying θ on both sides by μ, then changing t to t' on the left side and z to z' on the right side using Eq. [2]:

$$\partial\theta'/\partial t' = (\alpha^2/\delta)\partial\left[D(\theta)\partial\theta'/\partial z'\right]/\partial z' - (\mu\alpha/\delta)\left(\partial K/\partial\theta'\right)\left(\partial\theta'/\partial z'\right) \quad [3]$$

It is evident that Eq. [3] will have the same mathematical form as Eq. [2] only if the scale transformations:

$$\begin{aligned} D'\left(\theta'\right) &\equiv \mathrm{D}(\mu\theta) = (\alpha^2/\delta)D(\theta) \\ K'\left(\theta'\right) &\equiv \mathrm{K}(\mu\theta) = (\mu\alpha/\delta)K(\theta) \end{aligned} \quad [4]$$

apply to the two transport coefficients. Equation [4] connects values of the water diffusivity and the hydraulic conductivity at any two locations through scale factors. If this is a valid way to account for inherent spatial heterogeneity, the partial differential equation for $\theta(t, z)$ will indeed have the same mathematical form anywhere in a field. In this sense, the Richards equation and the physics which underlies it is said to be "scale-invariant." They are thus immune to the vagaries of inherent spatial heterogeneity as represented in the three scale factors (Eq. [2]) and the resultant scaling transformations of the two transport coefficients (Eq. [4]).

Scaling Soil Water Properties

Now, Eq. [4] cannot possibly be true for an arbitrary functional dependence of the two transport coefficients on θ. Determining precisely which functional relationships between water diffusivity and hydraulic conductivity and volumetric water content are compatible with Eq. [4] is quite important, because if the water content dependence of the transport coefficients is not consistent with the scaling of Eq. [1], the scale invariance of the differential equation will not be sustained. This critical issue can be investigated rigorously with methods in the mathematical theory of Lie groups, which show that, in order to be compatible with the scale invariance of the Richards equation, D(θ) and K(θ) each must have a power-law dependence on θ:

$$D(\theta) = a(\theta + b)^{\mathrm{m}} \quad [5]$$

$$K(\theta) = \left[ca^{\beta}/(1 + m\beta)\right](\theta + b)^{1+\mathrm{m}\beta} \quad [6]$$

where a, b, and c are constants determined by the units of D and K, the normalization of θ (e.g., dividing θ by its maximum value to create a water content variable ranging from 0 to 1), and the value of θ at which the two transport coefficients become equal to zero. The empirical parameters m and β are related to the scale factors in Eq. [2], as can be seen clearly after imposing Eq. [4] on Eqs. [5] and [6]:

$$D'\left(\theta'\right) \equiv a(\mu\theta + \mu b)^{\mathrm{m}} = \mu^{\mathrm{m}}D(\theta) = (\alpha^2/\delta)D(\theta) \quad [7]$$

$$K'\left(\theta'\right) \equiv \left[c\alpha^{\beta}/(1+m\beta)\right](\mu\theta+\mu b)^{1+m\beta} = \mu^{1+m\beta}K(\theta) = (\mu\alpha/\delta)K(\theta) \quad [8]$$

In Eqs. [7] and [8], the constant b is assumed to scale in the same way as θ does. It follows that:

$$\left(\alpha^2/\delta\right)^{1/m} = (\alpha/\delta)^{1/m\beta} \quad [9]$$

and, therefore:

$$\delta = \alpha^{(2\beta-1)/(\beta-1)} \quad [10]$$

Equations [7] and [8] then both yield the relationship:

$$\mu = \alpha^{1/m(1-\beta)} \quad [11]$$

for the scale factor assigned to θ. Thus, only the value of the scale factor α can be chosen independently, once m and β have been measured. Note also that $m \neq 0$, $m\beta \neq -1$ or 0, and $\beta \neq 1$ or 1/2. These three conditions are required in order that Eqs. [6], [9] and [10] be determinate.

A new facet of the scale factor α relating vertical coordinates at two locations emerges from Eq. [11] after invoking the definition of the water diffusivity:

$$D(\theta) \equiv \mathrm{K}(\theta)\frac{\partial\psi}{\partial\theta} \quad [12]$$

along with Eqs. [5] and [6] to derive a power-law expression for the matric potential ψ(θ) (L) that is consistent with scale invariance:

$$\psi(\theta) = [(1-m\beta)/cm(1-\beta)](\theta+b)^{m(1-\beta)} \quad [13]$$

The corresponding scaling transformation $\psi'(\theta') \equiv \psi(\mu\theta)$ then follows from Eq. [11]:

$$\psi'\left(\theta'\right) = [(1+m\beta)/cm(1-\beta)](\mu\theta+\mu b)^{m(1-\beta)} = \mu^{m(1-\beta)}\psi(\theta) = \alpha\psi(\theta) \quad [14]$$

Thus the scale factor α relates not only the vertical coordinate, but also the matric potential, at one place to that at another. Similarly, the scaling transformation of the hydraulic conductivity follows from Eqs. [4], [10] and [11] as:

$$K'\left(\theta'\right) = \alpha^{-\eta}K(\theta) \quad [15]$$

where

$$\eta \equiv (1+m\beta)/m(\beta-1) \quad [16]$$

is called the "Russo–Jury exponent."

In experimental studies of spatial heterogeneity, the right side of Eq. [15] is often expressed in terms of the square of a scale factor ω, which is defined by:

$$\omega \equiv \alpha^{2/\eta} \quad [17]$$

such that the scaling transformation of K(θ) takes on a single-parameter form:

$$K'\left(\theta'\right) = \frac{K(\theta)}{\omega^2} \quad [18]$$

Equations [14] and [18] together are known as the "Warrick–Nielsen scaling relationships." They have been tested in a large number of field-scale studies of the scaling of soil water behavior.

Testing Scale Invariance

The connection between Eqs. [6] and [13] and current field research in soil physics is facilitated after defining the constant b to equal $-\theta_r$, where θ_r is the "residual volumetric water content," a measure of the very small volume of water in soil that does not participate in flow on the time scale over which Eq. [1] is valid. Besides this prescription of b, a normalization of both θ and b by the difference, $(\theta_s - \theta_r)$, where θ_s is the volumetric water content associated with water saturation of the soil pores, a measure of soil porosity, is convenient in applications because it makes the volumetric water content automatically a scale-invariant quantity. With the two definitions:

$$\psi_s \equiv \psi(\theta_s),\ K_s \equiv K(\theta_s) \qquad [19]$$

Eqs. [6] and [13] now can be rewritten in the simple power-law forms:

$$\psi(S) = \psi_s S^{m(1-\beta)} \qquad [20]$$

$$K(S) = K_s S^{1+m\beta} \qquad [21]$$

where

$$S \equiv (\theta - \theta_r)/(\theta_s - \theta_r) \qquad [22]$$

is a normalized volumetric water content termed the relative saturation. Equations [20] and [21] show that this latter scale-invariant variable is a natural one with which to express the power-law relationship between matric potential or hydraulic conductivity and volumetric water content. Finally, Eqs. [20] and [21] also show that the hydraulic conductivity can be expressed directly as a function of matric potential:

$$K(\psi) = K_s(\psi_s/\psi)^{\eta} \qquad [23]$$

after noting Eq. [16]. The Russo–Jury exponent is thus revealed to be the parameter that determines a power-law dependence of hydraulic conductivity on the matric potential.

The power-law expression in Eq. [20] is widely used in soil physics to describe the functional relationship between matric potential and volumetric water content in field soils, where it is termed the "Brooks–Corey model":

$$S = (\psi_s/\psi)^{\lambda} \qquad [24]$$

Comparison with Eq. [20] shows that the Brooks–Corey exponent $\lambda = 1/m(\beta - 1)$. Representative values of the parameters ψ_s and λ have been calculated using hydraulic data on hundreds of soils grouped according to texture. These data show that ψ_s tends to increase, while the Brooks–Corey exponent λ tends to decrease, as the clay content of a soil becomes larger.

As an example of a controlled experiment testing Eqs. [20] and [21], **Table 1** lists field-wide mean values and coefficients of variation (CV, mean divided by standard deviation, a measure of spatial variability) for ψ_s, K_s, m, and β as deduced from measurements of $\psi(\theta)$ and $K(\theta)$ made at four soil profile depths in 0.8 ha field comprising an Alfisol (Hamra series) on which 30 instrumented plots were established. Notable in the table are the decline in K_s with depth, which is often observed, and the relatively large CV for this soil water parameter, as well as for m and β, thus indicating significant spatial variability. The lesser spatial variability reported for ψ_s parallels that observed for θ_s, for which the all-depth field-wide mean value was 0.37, CV = 0.11. The field-wide mean values of the Russo–Jury exponent η also tended to decline with depth: 4.3 ± 1.6, 3.3 ± 0.9, 2.9 ± 0.8, 2.9 ± 0.7. Its all-depth, field-wide mean value was found to be 3.3, CV = 0.36. Other studies on field soils have shown that η typically lies in the interval, 2.0–4.5, with CV in the range, 0.2–0.5.

The complete data set for the Hamra soil also was used to compute values of $\psi(S)$ and $K(S)$ at fixed values of the relative saturation, S, in the interval, 0.750–0.975, based on Eqs. [20] and [21]. The resulting all-depth, field-wide mean value and CV of the dimensionless ratios, $\psi(S)/\psi_s$ and $K(S)/K_s$, are listed in **Table 2** at 10 selected values of S. The very large coefficients of variation observed at smaller values of S signal just the kind of extreme spatial variability that the Warrick–Nielsen scale factors α (Eq. [14]) and ω (Eq. [18]) were defined to mediate under the basic premise of scale invariance. It should be possible to substitute the all-depth mean values of $\psi(S)/\psi_s$ and $K(S)/K_s$ into the left sides of Eqs. [14] and [18] to calculate α and ω values at locations anywhere in the field, after imposing an all-depth mean value of 1.0 on each scale factor, since only the ratios $\psi(S)/\psi_s$ and $K(S)/K_s$ are being used to calculate them. The results are summarized in **Table 3**, which shows that α and ω indeed do have the same average spatial variability over the field and that the field-wide mean value of η calculated from the two scale factors using Eq. [17] is exactly the same as the all-depth, field-wide mean value of η cited above, which was calculated directly from measurements of the matric

Table 1 Field-wide mean value and coefficient of variation (CV) for the scaling parameters, ψ_s, K_s, m, and β, based on field data for the Hamra soil, Bet Dagan, Israel

	ψ_s (m)		K_s (10^{-5} m s^{-1})		m		β	
Depth (m)	*Mean*	*CV*	*Mean*	*CV*	*Mean*	*CV*	*Mean*	*CV*
0.0	−0.072	0.22	6.12	0.41	1.86	0.68	1.46	0.68
0.3	−0.072	0.23	4.47	0.36	2.50	0.65	1.60	0.65
0.6	−0.074	0.21	2.17	0.65	3.19	0.82	1.69	0.82
0.9	−0.082	0.27	1.98	0.75	3.35	0.81	1.70	0.81
0.0–0.9	−0.074	0.24	3.66	0.69	2.48	0.88	1.60	0.88

m, β, empirical parameters related to scale factors in Eq. [2].
ψ_s and K_s, "air-entry" matric potential and saturated hydraulic conductivity, respectively.
Source: Adapted from Russo D and Bresler E (1981) Soil hydraulic properties as stochastic processes. *Soil Science Society of America Journal* 45: 682–687.

Table 2 All-depth, field-wide mean values and coefficients of variation (CV) for the relative matric potential ($\psi(S)/\psi_s$) and hydraulic conductivity ($K(S)/K_s$) in the Hamra soil

S	$\psi(S)/\psi_s$	*CV*	$K(S)/K_s$	*CV*
0.750	25	3.9	0.14	0.92
0.775	13	3.3	0.17	0.86
0.800	7.7	2.8	0.20	0.80
0.825	4.8	2.2	0.24	0.74
0.850	3.3	1.6	0.28	0.68
0.875	2.4	1.2	0.34	0.61
0.900	1.9	0.8	0.41	0.53
0.925	1.5	0.5	0.50	0.45
0.950	1.3	0.3	0.61	0.34
0.975	1.1	0.13	0.77	0.20

Source: Adapted from Jury WA, Russo D, and Sposito G (1987) The spatial variability of water and solute transport properties in unsaturated soil. II. Scaling models of water transport. *Hilgardia* 55: 33–36.

Table 3 All-depth, field-wide mean values and coefficients of variation (CV) for the scale factors α and ω along with the Russo–Jury exponent (Eq. [17])

Parameter	*Mean*	*CV*
α	1.00[a]	0.435
ω	1.00[a]	0.446
η	3.36	0.354

[a]Unit normalization imposed.
Source: Adapted from Jury WA, Russo D, and Sposito G (1987) The spatial variability of water and solute transport properties in unsaturated soil. II. Scaling models of water transport. *Hilgardia* 55: 33–56.

potential and hydraulic conductivity. These results demonstrate that the spatial variability of the two soil water properties in the Hamra soil is accounted for quite well by using the Warrick–Nielsen scaling factors. At each location in the field, the scaling factors for the location can be used along with field-wide mean values of the soil water properties to estimate local values of these properties, and the solution of the field-wide average Richards equation can be transformed with the scale factors into a local solution of the equation.

Why Scaling Matters

What insights does the scaling of soil physical properties and processes provide? The basic premise is that the partial differential equation governing the behavior of the properties and processes is scale-invariant. The example in the present case is the Richards equation for one-dimensional infiltration (Eq. [1]), which describes volumetric water content changes in space and time when water moves downward in a soil profile. Scale invariance of this partial differential equation implies that it remains valid from place to place in a field soil, with the same functional relationships between its two transport coefficients, water diffusivity and hydraulic conductivity, and the volumetric water content anywhere. The transport coefficients may fluctuate in value wildly across a field because of inherent spatial heterogeneity resulting from varying texture and morphology, but they are none the less connected by scale factors that relate their values between any two places. Their spatial variability is thus accounted for solely by differences in the local scales of volumetric water content, position, and time (Eq. [2]) that prevail at different locations, or, if the relative saturation (Eq. [22]) is the dependent variable in the Richards equation, solely by differences in the scales of position and time that attend water movement at different locations.

Implicit in the scale invariance of the Richards equation, as noted in connection with the Hamra soil study, is the premise that it must also apply on average to an entire field. This condition can be expressed in the field-wide mean Richards equation:

$$\partial S/\partial t = \partial[D_m(S)\partial S/\partial z]/\partial z - (\partial K_m/\partial S)(\partial S/\partial z) \quad [25]$$

where $D_m(S)$ and $K_m(S)$ are field-wide mean values of the transport coefficients. Since the dependent variable in Eq. [25] is automatically scale-invariant, and the mean transport coefficients are related to their local values at any place "P" through the Warrick–Nielsen scaling relationships:

$$\alpha_P^2 D_P(S)/\omega_P^2 = D_m(S) \quad [26a]$$

$$K_P(S)/\omega_P^2 = K_m(S) \quad [26b]$$

with a corresponding equation for matric potential:

$$\alpha_P \psi_P(S) = \psi_m(S) \qquad [26c]$$

Equation [26a] follows from Eqs. [12], [14] and [18]. In field studies, ψ_m and K_m are first calculated, then used along with local values of the matric potential and hydraulic conductivity to estimate a set of α_P and ω_P values, thus giving substance to the reality of Eq. [25]. Using this approach, one need only solve the mean Richards equation for a chosen boundary-value problem and have in hand a set of scale factors $\{\alpha_P, \omega_P\}$ in order to apply the solution at every point in a field — clearly an advantage in predictive applications.

The compatibility conditions in Eqs. [5], [6] and [13] ensure that the scale invariance of the Richards equation is not a vacuous hypothesis: The matric potential and the two transport coefficients are required to be power-law functions of the volumetric water content or, equivalently, the relative saturation. A power-law relationship is also expected between hydraulic conductivity and and matric potential (Eq. [23]), if the Richards equation is scale-invariant. This relationship is often observed, but other models of $K(\psi)$ have been applied in soil physics research. For example, the well-known Gardner model postulates an exponential relationship for $K(\psi)$:

$$K(\psi) = K_S \exp\left[-\frac{\psi_s - \psi}{D_s/K_s}\right] \qquad [27]$$

If Eq. [27] accurately describes the behavior of the hydraulic conductivity of a soil, the Richards equation incorporating it will not be scale-invariant under the scale transformations in Eq. [2]. Indeed, verification of the scaling relations in Eqs. [14] and [15] becomes impossible. The underlying physical reason for this problem is signaled by the appearance of the ratio, D_s/K_s, in Eq. [27]. This ratio has dimensions of length and determines how quickly $K(\psi)$ declines as ψ decreases below its value at water-saturation in a "Gardner soil." It is, therefore, an intrinsic length scale in the Gardner model, determined uniquely by the values of D_s and K_s at each location in a field. Using the methods of the mathematical theory of Lie groups, it can be shown that Eq. [27] is the result of setting $m = -2$ in Eq. [5], known as the "Fujita model" of the water diffusivity, while setting $\beta = 1$, which is not permitted in connection with the derivation of Eqs. [9]–[11]. Scale invariance of the Richards equation can none the less be maintained if $\alpha \equiv 1$, leaving only time and volumetric water content to be scaled, in which case Lie group methods then show that Eq. [27] specifies $K(\psi)$. Thus the Gardner model results from breaking the full space-time scaling symmetry of the Richards equation to eliminate the vertical coordinate and allow for an intrinsic length scale.

In terms of the conventional water-desorption methods of measuring the cumulative pore-size distribution in a soil by combining data on $\psi(S)$ with the Young–Laplace equation to calculate an effective pore radius as a function of relative saturation, the length scale D_s/K_s can be interpreted as characterizing the size of the very first pores to empty when ψ drops below ψ_s. The existence of such an intrinsic length scale thus contradicts the use of an arbitrary scale factor for relating matric potential at two places in a field (Eq. [14]). This example of broken scaling symmetry is valuable, in that it illuminates by contrast the underlying physical significance of Eqs. [14]–[18]: that the Richards equation operates everywhere in a field without prejudice as to the local scales of length, time, or water content.

Further Reading

Assouline S and Or D (2013) Conceptual and parametric representation of soil hydraulic properties: A review. *Vadose Zone Journal* 12: 20. http://dx.doi.org/10.2136/vzj2013.07.0121.
Ghanbarian-Alavijeh B, Millán H, and Hunag G (2011) A review of fractal, prefractal and pore-solid-fractal models for parameterizing the soil water retention curve. *Canadian Journal of Soil Science* 91: 1–14.
Hillel D and Elrick DE (eds.) (1990) *Scaling in soil physics*. Madison, WI: Soil Science Society of America.
Pachepsky Y, Radcliffe DE, and Selim HM (eds.) (2003) *Scaling methods in soil physics*. Boca Raton, FL: CRC Press.
Sposito G (1990) Lie group invariance of the Richards equation. In: Cushman JH (ed.) *Dynamics of fluids in hierarchical porous media*, pp. 327–347. San Diego, CA: Academic Press.
Sposito G (1995) Recent advances associated with soil water in the unsaturated zone. *Reviews of Geophysics* 33(Suppl.2): 1059–1065.
Sposito G (ed.) (1998) *Scale dependence and scale invariance in hydrology*. Cambridge, UK: Cambridge University Press.
Warrick AW (2003) *Soil water dynamics*. New York: Oxford University Press.